Lecture 25

11.5 - Alternating Series

The tests we dealt with so far only allow us to deal with series with all positive terms (we can use them to deal with series with all negative terms too, by factoring out the -1), but what do we do with series with both positive and negative terms?

Def: An alternating series is a series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^n b_n$$

where $b_n > 0$ for all n.

Ex: $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(\frac{\pi}{3}\right)^{2n-1}}{(2n-1)!}$$

are alternating series.

Alternating Series Test

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ $\left(\text{or } \sum_{n=1}^{\infty} (-1)^n b_n \right)$

satisfies (i)

(ii)

then the series converges.

(**Note**: The usual divergence test applies to alternating series, indeed any series.)

proof:

Ex: Determine whether the following series converge:

ⓐ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ⓑ $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$ ⓒ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$

ⓓ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{2n+3}$ ⓔ $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{2\pi}{n}\right)$

Estimating the Error

Suppose $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges to s. That is, the sequence of partial sums $s_n = b_1 - b_2 + b_3 - \cdots + (-1)^{n-1}b_n$ converges to s. Because of the way alternating series converge, it's pretty simple to estimate the sum and get an error bound, $|R_n| = |s - s_n|$

Alternating Series Estimation Theorem

If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$, $b_n > 0$ such that

(i) $b_{n+1} < b_n$ for all n & (ii) $\lim_{n\to\infty} b_n = 0$

then
$$|R_n| = |s - s_n| \le b_{n+1}.$$

Proof:

Ex: How many terms of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$$

do we have to add up to approximate the sum to within 10^{-5}?